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Gravitoelectromagnetic approach to the gravitational Faraday rotation in stationary spacetimes

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Using the $1+3$ formulation of stationary spacetimes we show, in the context of gravitoelectromagnetism, that the plane of the polarization of light rays passing close to a black hole undergoes a rotation. We show that this rotation has the same integral form as the usual Faraday effect; i.e., it is proportional to the integral of the component of the gravitomagnetic field along the propagation path. We apply this integral formula to calculate the Faraday rotation induced by the Kerr and NUT spaces using the quasi-Maxwell form of the vacuum Einstein equations. [S0556-2821(99)00214-3]

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I. INTRODUCTION

It is a well-known fact that the plane of the polarization of light rays passing through plasma in the presence of a magnetic field undergoes a rotation which is called a Faraday rotation (Faraday effect) [1]. One can show that a plane-polarized wave is rotated through an angle $\Delta\theta$ given by

$$\Delta\theta = \frac{2\pi e^3}{m^2 c^2 \omega^2}\int_a^b n B_{||} dl, \qquad (1)$$

where $B_{||}$ is the component of the magnetic field along the line of sight.

It is also a well-known consequence of general relativity that light rays passing a massive object are bent towards it. Several authors have considered the gravitational effect on the polarization of light rays by analogy with the Faraday effect [2,3]. In particular they have considered the propagation of electromagnetic waves in Kerr spacetime. In [3] the authors have used the Walker-Penrose constant to calculate this effect for a Kerr black hole. They have shown that in the weak field limit the rotation angle of the plane of the polarization is proportional to the line-of-sight component of the black hole's angular momentum at third order. In what follows we will use the Landau-Lifshitz $1+3$ splitting of stationary spacetimes and show that the gravitational Faraday rotation has the same integral form as the usual Faraday effect if one replaces the magnetic field with the gravitomagnetic field of the spacetime under consideration. Having found this integral form, one can use the quasi-Maxwell form of the vacuum Einstein equations to calculate the effect much more easily. In particular we show that the gravitational Faraday rotation in Newman-Unti-Tamburino (NUT) space is zero, a result which needs a lot of calculation if one uses an approach based on the Walker-Penrose constant.

II. 1+3 FORMULATION OF STATIONARY SPACETIMES (PROJECTION FORMALISM)

Suppose that $\mathcal{M}$ is the four-dimensional manifold of a stationary spacetime with metric[1] g_{ab} and $p\in\mathcal{M}$; then one can show that there is a three-dimensional manifold Σ_3 defined invariantly by the smooth map [4]

$$\Psi:\mathcal{M}\rightarrow\Sigma_3,$$

where $\Psi=\Psi(p)$ denotes the orbit of the timelike Killing vector $\boldsymbol{\xi}_t$ passing through p. The three-space Σ_3 is called the factor space $\mathcal{M}/G_1$, where G_1 is the one-dimensional group of transformations generated by $\boldsymbol{\xi}_t$. Using a coordinate system adapted to the congruence $\xi_t=\partial_t$ we denote the projected three-dimensional metric on Σ_3 by $\gamma_{\alpha\beta}$ $(\alpha,\beta=1,2,3)$. These are the coordinates comoving with respect to the timelike Killing vector. One can use $\gamma_{\alpha\beta}$ to define differential operators on Σ_3 in the same way that g_{ab} defines differential operators on $\mathcal{M}$. For example the covariant derivative of a three-vector $\mathbf{A}$ is defined as

$$A^{\alpha}_{;\beta}=\partial_\beta A^\alpha+\lambda^{\alpha}_{\gamma\beta}A^\gamma,$$

$$A_{\alpha;\beta}=\partial_\beta A_\alpha-\lambda^{\gamma}_{\alpha\beta}A_\gamma,$$

where $\lambda^{\alpha}_{\gamma\beta}$ is the three-dimensional Christoffel symbol cosntructed from the components of $\gamma_{\alpha\beta}$ in the following way:

$$\lambda^{\sigma}_{\mu\nu}=\frac{1}{2}\gamma^{\sigma\eta}(\partial_\nu\gamma_{\eta\mu}+\partial_\mu\gamma_{\eta\nu}-\partial_\eta\gamma_{\mu\nu}).$$

The metric of a stationary spacetime can be written in the following form [5]:

$$ds^2=h(dx^0-A_\alpha dx^\alpha)^2-dl^2, \qquad (2)$$

where

[1]Note that the Roman indices run from 0 to 3 and Greek indices from 1 to 3.

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$$A_\alpha \equiv g_\alpha = \frac{-g_{0\alpha}}{g_{00}}, \quad h \equiv g_{00},$$

and

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta = \left(-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}} \right) dx^\alpha dx^\beta$$

is the spatial distance written in terms of the three-dimensional metric $\gamma_{\alpha\beta}$ of Σ_3. Using this formulation for a stationary spacetime one can write the vacuum Einstein equations in the following quasi-Maxwell form [6]:

$$\text{div } \mathbf{B}_g = 0, \tag{3}$$

$$\text{Curl } \mathbf{E}_g = 0, \tag{4}$$

$$\text{div} \mathbf{E}_g = -[\tfrac{1}{2}(\sqrt{h} B_g)^2 + E_g^2], \tag{5a}$$

$$\text{Curl}(\sqrt{h} \mathbf{B}_g) = 2 \mathbf{E}_g \times (\sqrt{h} \mathbf{B}_g), \tag{5b}$$

$$P^{\alpha\beta} = E_g^{\alpha;\beta} + [(\sqrt{h} B_g^\alpha)(\sqrt{h} B_g^\beta) - (\sqrt{h} B_g)^2 \gamma^{\alpha\beta}] + E_g^\alpha E_g^\beta, \tag{6}$$

where the gravitoelectromagnetic fields are

$$\mathbf{E}_g = -\nabla \ln h^{1/2} = -\frac{1}{2} \frac{\boldsymbol{\nabla} h}{h}, \tag{7}$$

$$\mathbf{B}_g = \text{Curl } \mathbf{A}, \tag{8}$$

and $P^{\alpha\beta}$ is the three-dimensional Ricci tensor constructed from the metric $\gamma_{\alpha\beta}$. It is attractive to regard the combination $\sqrt{h} B_g$, appearing in the above equations, as the gravitational analogue of the magnetic intensity field **H** and denote it by $\mathbf{H}_g$. In this way one may think of the right hand side (RHS) of Eq. (5b) as an energy current corresponding to a Poynting vector flux of gravitational field energy. Note that all operations in these equations are defined in the three-dimensional space with metric $\gamma_{\alpha\beta}$. Using the timelike Killing vector of the spacetime one can define the above gravitoelectromagnetic fields in the following covariant forms:

$$E_g^b = -\frac{1}{2} \frac{(\xi^a \xi_a)^{;b}}{|\xi|^2}, \quad |\xi| = h^{1/2}, \tag{9}$$

$$B_g^b = -\frac{1}{2} |\xi| \xi^a \varepsilon_a^{\ bcd} \left[\left(\frac{\xi_d}{|\xi|^2} \right)_{;c} - \left(\frac{\xi_c}{|\xi|^2} \right)_{;d} \right], \tag{10}$$

where $\varepsilon_a^{\ bcd}$ is the four-dimensional antisymmetric tensor and the semicolon denotes covariant differentiation.

III. DERIVATION OF THE GRAVITATIONAL FARADAY ROTATION

Using the analogy with the flat spacetime we take the plane of the polarization of an electromagnetic wave to consist of two three-vectors **k** and **f**, the wave vector, and the polarization vector, respectively. The four-vectors corresponding to these two three-vectors have the following relations:

$$k^a k_a = 0, \quad k^a f_a = 0, \quad f^a f_a = 1, \quad a = 0,1,2,3. \tag{11}$$

Both of these four-vectors are parallelly transported along null geodesics [5], i.e.,

$$\frac{\partial k^a}{\partial \lambda} + \Gamma^a_{mn} k^n k^m = 0, \tag{12}$$

$$\frac{\partial f^a}{\partial \lambda} + \Gamma^a_{mn} f^n k^m = 0, \tag{13}$$

where λ is an affine parameter varying along the ray. Employing an orthogonal decomposition based on the adapted coordinates, the above three-vectors defined on the three-space Σ_3 can be taken to be equivalent to the contravariant components of k^a and f^a, i.e., $\mathbf{k} \equiv {}^{(3)}k^\alpha = {}^{(4)}k^\alpha$ and $\mathbf{f} \equiv {}^{(3)}f^\alpha = {}^{(4)}f^\alpha$ [7]. One should note that the covariant counterparts of these three-vectors are not the spatial components of the covariant four-vectors k_a and f_a but

$${}^{(3)}k_\beta = \gamma_{\alpha\beta} {}^{(3)}k^\alpha = {}^{(4)}k_\beta + k_0 g_\beta$$

and

$${}^{(3)}f_\beta = \gamma_{\alpha\beta} {}^{(3)}f^\alpha = {}^{(4)}f_\beta + f_0 g_\beta .$$

From Eq. (11) one can see that the polarization vector is known up to a constant multiple of the wave vector; i.e., both f_a and $f'_a = f_a + C k_a$ satisfy Eq. (11). This shows that there is a kind of gauge freedom in choosing f which enables one to put $f_0 = 0$ without loss of generality and in which case ${}^{(3)}f_\beta = {}^{(4)}f_\beta$.[2] Applying the above decomposition and equations of parallel transport (12) and (13), Fayos and Llosa [2] have arrived at the following two equations for the evolution of the three-vectors **k** and **f** along the ray:

$${}^3\nabla_{\mathbf{k}} \mathbf{k} = \mathbf{L} \times \mathbf{k} + (\mathbf{E}_g \cdot \mathbf{k}) \mathbf{k}, \tag{14}$$

$${}^3\nabla_{\mathbf{k}} \mathbf{f} = \mathbf{L} \times \mathbf{f}, \tag{15}$$

where

$$\mathbf{L} = -\frac{1}{2} k_0 \left[\mathbf{B}_g - \frac{1}{2} (\mathbf{B}_g \cdot \mathbf{f}) \mathbf{f} + \frac{1}{|\mathbf{f}|} \mathbf{E}_g \cdot (\mathbf{k} \times \mathbf{f}) \mathbf{f} \right]. \tag{16}$$

Note that we have written their results in terms of the gravitoelectromagnetic fields defined in Eqs. (7) and (8). If we had only the second term on the RHS of Eq. (14), that would have meant, by comparison with the four-dimensional definition of the parallel transport, that the three-dimensional vecor **k** is parallely transported along the projection of the null geodesic in Σ_3 space. But the appearance of the first

[2] This choice corresponds to $C = -f_0/k_0$ and makes f orthogonal to the time lines.

term shows that **k** has also been rotated by an angular velocity **L**. The same rotation happens to the polarization vector **f** as can be seen from Eq. (15). Therefore the combination of these two equations leads to the fact that the polarization plane has rotated by angular velocity **L** along the projected null geodesic. The angle of rotation around the tangent vector $\hat{\mathbf{k}}$ along the path between the source and the observer is given by

$$\Omega=\int_{sou}^{obs}\mathbf{L}\cdot\hat{\mathbf{k}}d\lambda=-\frac{1}{2}\int_{sou}^{obs}k_0\mathbf{B}_g\cdot\hat{\mathbf{k}}d\lambda, \tag{17}$$

where we used the fact that $\mathbf{f}\cdot\mathbf{k}=0$, which follows from Eq. (11) and the choice $f_0=0$. Now combining the two equations

$$k_0=g_{0a}k^a=h(k^0-g_\alpha k^\alpha),$$

$$k^ak_a=0\equiv h(k^0-g_\alpha k^\alpha)^2-\gamma_{\alpha\beta}k^\alpha k^\beta=0,$$

we have

$$\frac{k_0^2}{h}-\gamma_{\alpha\beta}k^\alpha k^\beta=0 \tag{18}$$

or, equivalently, in terms of $k^\alpha=dx^\alpha/d\lambda$,

$$\frac{k_0^2}{h}=\left(\frac{dl}{d\lambda}\right)^2. \tag{19}$$

Finally upon substitution of Eq. (19) in Eq. (17) and putting $\hat{\mathbf{k}}dl=\mathbf{dl}$ we find

$$\Omega=-\frac{1}{2}\int_{sou}^{obs}\sqrt{h}\mathbf{B}_g\cdot\mathbf{dl}, \tag{20}$$

which has the same integral form as Eq. (1); i.e., the gravitational Faraday rotation is proportional to the integral of the component of the gravitomagnetic field along the propagation path. But their main difference is the fact that the gravitational Faraday rotation, given by Eq. (20), is a purely geometrical effect while the usual Faraday effect, Eq. (1), depends on the frequency of the light ray. In the next two sections we will apply this formula to the cases of NUT and Kerr black holes.

IV. GRAVITATIONAL FARADAY ROTATION IN NUT SPACE

There is no gravitational Faraday rotation induced by NUT space and the reason is as follows. Take a closed path $\mathcal{C}$ around the NUT hole which consists of two paths (see Fig. 1): path 1, a null geodesic which passes close to the black hole, and path 2, so far away that the effect of the gravitational field (including the Faraday rotation) on the light rays is negligible (another reason that on path 2 there is no gravitational Faraday rotation is the fact that $\mathbf{B}_g\to 0$ as $r\to\infty$). Now using the Stokes theorem, one can write Eq. (20) in the form

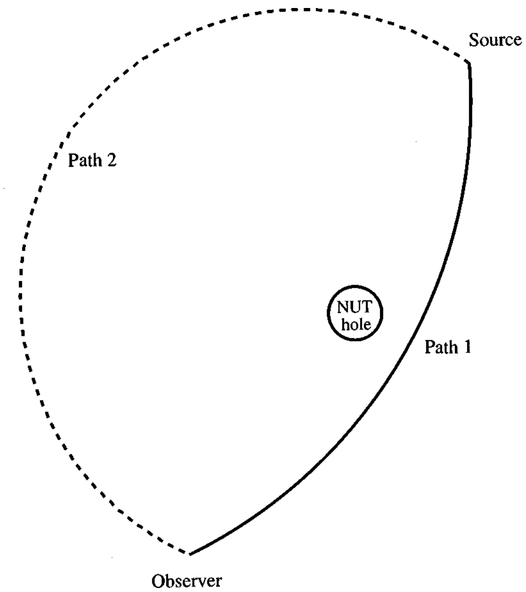

FIG. 1. The NUT hole and a closed path around it.

$$\Omega=-\frac{1}{2}\oint_{\mathcal{C}}(\sqrt{h}\mathbf{B}_g)\cdot\mathbf{dl}=-\frac{1}{2}\int_s\nabla\times(\sqrt{h}\mathbf{B}_g)\cdot\mathbf{dS},$$

and using Eq. (5b) we have

$$-\frac{1}{2}\int_1(\sqrt{h}\mathbf{B}_g)\cdot\mathbf{dl}-\frac{1}{2}\int_2(\sqrt{h}\mathbf{B}_g)\cdot\mathbf{dl}=-\int_s(\mathbf{E}_g\times\sqrt{h}\mathbf{B}_g)\cdot\mathbf{dS}. \tag{21}$$

The second term on the LHS of the above equation is zero by costruction. On the other hand for the NUT space we have[3]

$$\mathbf{E}_g=-\frac{1}{2}\partial_r[\ln f(r)]\hat{\mathbf{r}}$$

and

$$\mathbf{B}_g=\frac{2lf(r)^{1/2}}{r^2}\hat{\mathbf{r}},$$

which together show that the RHS of Eq. (21) is also zero and therefore

$$-\frac{1}{2}\int_1(\sqrt{h}\mathbf{B}_g)\cdot\mathbf{dl}=0; \tag{22}$$

[3]We have used the following form of the NUT metric:

$$ds^2=f(r)(dt-2l\cos\theta d\phi)^2-f(r)^{-1}dr^2-(r^2+l^2)(d\theta^2+\sin^2\theta d\phi^2),$$

where

$$f(r)=1-\frac{2(mr+l^2)}{(r^2+l^2)}.$$

i.e., there is no Faraday effect on the light rays passing a NUT black hole close by. One can show that the same result can be obtained for the NUT space using the approach based on the Walker-Penrose constant. In this case one needs to take into account the simplifying fact that all the geodesics in NUT space including the null ones lie on spatial cones [6].

V. GRAVITATIONAL FARADAY ROTATION IN KERR METRIC

The Faraday effect in a Kerr metric has already been studied and it has been shown that despite previous claims [8], when a light ray passes through the vacuum region outside rotating matter its polarization plane rotates [3]. In this section we will consider two different cases: (1) when the orbit lies in the equatorial plane, i.e., for $\theta=\pi/2$; (2) a more general orbit which intersects the equatorial plane and is symmetric about it.

A. Orbits in the equatorial plane

In this case using the definitions of $\mathbf{E}_g$ and $\mathbf{B}_g$ and Eq. (21) one can see that the gravitational analogue of the Poynting vector defined by $\mathbf{E}_g\times\sqrt{h}\mathbf{B}_g$ has only one component along the ϕ direction and therefore is normal to the plane of the orbit, which in turn leads to the fact that in this special case there is no gravitaional Faraday effect on light rays.

B. Symmetric orbit about the equatorial plane

In this case we need to find the orbit and we will see that one just needs to find the orbit in the zeroth order in a/r and m/r (i.e., straight line approximation) which is done in the Appendix.

Writing the Kerr metric in the form (2) in Boyer-Lindquist coordinates one can see that

$$\mathbf{A}=A_\phi=\frac{2amr\sin^2\theta}{2mr-\rho^2},$$

from which we have

$$B_g^r=\frac{2amr\sin 2\theta[2mr-r^2-a^2]}{\sqrt{\gamma}(2mr-r^2-a^2\cos\theta^2)}$$

and

$$B_g^\theta=\frac{2am\sin^2\theta(a^2\cos\theta^2-r^2)}{\sqrt{\gamma}(2mr-r^2-a^2\cos^2\theta)^2},$$

where

$$\gamma=\det\gamma_{\alpha\beta}\quad\text{and}\quad\rho^2=r^2+a^2\cos^2\theta.$$

Using the definition of the gravitoelectric field given in Eq. (7) we have

$$E_g^r=\frac{\Delta m(a^2\cos^2\theta-r^2)}{\rho^4(\rho^2-2mr)}$$

and

$$E_g^\theta=\frac{rma^2\sin 2\theta}{\rho^4(\rho^2-2mr)},$$

where $\Delta=r^2+a^2-2mr$. Substituting the above fields in Eq. (21) and putting $\mu=\cos\theta$ we have

$$\Omega=-\int_s(\mathbf{E}_g\times\sqrt{h}\mathbf{B}_g)_\phi\mathbf{dS}^\phi$$
$$=2am^2\int_{-\mu_0}^{\mu_0}\int_{r_{orb}(\mu)}^{\infty}\frac{drd\mu}{(r^2+a^2\mu^2-2mr)^2},\qquad(23)$$

where $r_{orb}(\mu)$ is the equation (of the projection) of the orbit in the (r,θ) plane. To find the lowest order Faraday effect we calculate the above integral neglecting the a^2/r^2 and m/r terms in which case we have

$$\Omega=2am^2\int_{-\mu_0}^{\mu_0}\int_{r_{orb}(\mu)}^{\infty}\frac{1}{r^4}\mathrm{d}r\mathrm{d}\mu=-\frac{4}{3}am^2\int_0^{\mu_0}\frac{1}{r_{orb}^3}\mathrm{d}\mu.$$

Using the (r,θ) equation of the orbit given in the Appendix one can calculate the above integral which gives

$$\Omega=-\frac{4}{3}am^2\int_{\mu_0=\sqrt{\eta}/r_{min}}^{0}\left(1-\frac{r^2{}_{min}}{\eta}\mu^2\right)^{3/2}d\mu$$
$$=(1/4)\pi\cos\theta_0\frac{am^2}{r^3{}_{min}}.$$

This expression is of third order $am^2/r^3{}_{min}$ which is of the same order as the result given in [3].

VI. DISCUSSION

We have shown that using the 1+3 formulation of stationary spacetimes one can cast the gravitational Faraday rotation in exactly the same mathematical form as the usual Faraday effect; i.e., the gravitational Faraday rotation is proportional to the gravitomagneic field of the spacetime along the propagation path of the light ray. One should note that the origins of these two effects are completely different. The usual Faraday effect originates from the interactions between the electrons in a plasma on the one hand with the electromagnetic field of the light ray and the external magnetic field on the other hand and therefore depends on the frequency of the light ray. But the gravitational Faraday effect is a purely geometrical one originating from the structure of the spacetime under consideration and it is normally attributed to the behavior of the reference frames outside the spacetime of a rotating body. Using the quasi-Maxwell form of the vacuum Einstein equations we showed that there is an easy way to calculate the effect by transforming the line integral of $\mathbf{B}_g$ to a surface integral of the gravitational analogue of the Poynting vector. More importantly the order of the effect can be seen without going through the detailed calculation [as in Eq. (23) for the Kerr case] and in some cases like NUT space just a simple observation reveals that there is no effect at all. For gravitational waves of small amplitude propagating in a curved background, one can develop the geometric optics in such a way that the wave and polarization four-vectors satisfy the same relations as given by Eqs. (11) [9]. So it can easily be seen that all the main relations that we have found for the gravitational Faraday rotation of light rays are also applicable to gravitational waves of small amplitude.

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APPENDIX

The equation governing the projection of the orbit in the (r,θ) plane for Kerr metric is given by [10]

$$\int^{r} \frac{dr}{\sqrt{r^4+(a^2-\xi^2-\eta)r^2+2m[\eta+(\xi-a)^2]r-a^2\eta}} = \int^{\theta} \frac{d\theta}{\sqrt{\eta+a^2\cos^2\theta-\xi^2\cot^2\theta}}, \tag{A1}$$

where ξ and η are constants of the motion and we choose the case in which $\eta>0$, which corresponds to the null geodesics which intersect the equatorial plane and are symmetric about it [10]. We perform the above integrations for the case when $a/r \ll 1$ and $m/r \ll 1$ i.e., for weak deflections, and indeed as we will see for a case in which there is no deflection in the (r,θ) plane. First we evaluate the LHS of the above equation which can be written in the following form (after discarding the small terms):

$$\int dr/r^2\sqrt{1-r^2{}_{min}/r^2} = (1/r_{min})\arccos(r_{min}/r), \tag{A2}$$

where $r_{min}=\sqrt{\xi^2+\eta}$ is the leading term (in the expansion) of the largest root of $r^4+(a^2-\xi^2-\eta)r^2+2m[\eta+(\xi-a)^2]r-a^2\eta=0$ for small deflection [3].

Now we evaluate the RHS of Eq. (A1) in the same limit. This integral can be written in the following form:

$$\mathrm{RHS} = -\int \frac{d\mu}{\sqrt{\eta+\mu^2(a^2-\xi^2-\eta)-a^2\mu^4}}.$$

Now using the fact that $a/r_{min} \ll 1$ and $r_{min}=\sqrt{\xi^2+\eta}$ we can approxiamte and evaluate the above integral as follows:

$$\mathrm{RHS} = -\int \frac{d\mu}{\sqrt{\eta-\mu^2 r^2{}_{min}}} = -(1/r_{min})\arcsin(\mu\sqrt{r^2{}_{min}/\eta}). \tag{A3}$$

Equating Eqs. (A2) and (A3) we have

$$r_{orb} = \frac{r_{min}}{\sqrt{1-(r^2{}_{min}/\eta)\cos^2\theta}}, \tag{A4}$$

which is the projection of the orbit in the (r,θ) plane for small deflections and in this case in fact no deflection because there is no term depending on m or a. As one can see $r\to\infty$ when $\cos\theta=\pm\sqrt{\eta}/r_{min}$ where plus and minus signs correspond to the position angles θ_o and θ_s of the observer and the source, respectively.

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